PERGAMON

Applied Mathematics Letters 16 (2003) 127–130

Applied Mathematics Letters

www.elsevier.com/locate/aml

Limit Cycles in an Optimal Control Problem of Diabetes

J. R. FARIA*
School of Finance and Economics
University of Technology, Sydney, Australia
and
Department of Economics, University of New Hampshire
Durham, NH 03824-3593, U.S.A.
jfaria@unh.edu

(Received August 2000; revised and accepted March 2001)

Abstract—This article examines the behaviour of an individual diagnosed with diabetes. It is shown that the medical treatment of the disease creates incentives that make a diabetic's consumption, weight, and labour supply display cyclical patterns. The existence of a limit cycle is proved using an adaptation of the Hopf bifurcation theorem for optimal control problems. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Limit cycles, Hopf bifurcation theorem.

1. INTRODUCTION

Diabetes is a chronic disease that affects around 5% of the U.S. population. It is the seventh leading cause of death in the United States. The treatment of diabetes is based on three principles. The diabetic has to take medications (such as insulin), follow a strict diet, and practice physical exercises [1]. Overall, the medical treatment has welfare costs associated with labour supply and consumption habits.

However, the last few decades have seen the rise of new and powerful drugs and medical treatments, and the proliferation of new food products with low sugar and cholesterol which impact the diabetic behaviour. As a result of these medical and food innovations, on the one hand, the diabetic may be "lulled" into thinking that he does not have to follow strictly the whole treatment (on lulling effect, see [2]). That is, he has an incentive to do fewer physical exercises or to relax in his diet. On the other hand, if the diabetic does not follow the medical treatment accordingly, he may incur health problems with related welfare costs. These may induce the diabetic to adhere to diet and exercise.

This paper shows how the incentives concerning the medical treatment of diabetes can lead to a cyclical behaviour in the weight and consumption habits of the diabetic (for related literature, see [3–5]). One important consequence of the cyclical pattern in consumption is that labour supply can be cyclical as well. The model is general enough to be adapted to other diseases,

*Address for correspondence is the second listed address.

0893-9659/02/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved. Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0893-9659(02)00154-4

which similarly to diabetes require continuous medical treatment, adherence to strict diet, and practice of physical exercise, such as heart diseases.

2. THE MODEL

The representative consumer is a diagnosed diabetic. He has to follow a medical treatment (T), maintain a strict diet and exercise routine. Medical treatment has three welfare effects, one positive and the other two negative. The utility function that captures the these welfare effects is

$$U(c,T) = \ln c - v\,l(T) - \Omega(T), \tag{1}$$

where the first term of the right-hand side is the overall improvement in health. The health state is assumed to be a function of the consumption pattern (c) of the diabetic. The second term is the impact of medical treatment on labor supply (l). Finally, the third term describes the direct costs of medical treatment in welfare terms, that is, the constraints implied by strict rules for diet, exercise, etc.[1]

The medical treatment is assumed to decrease the supply of labour. A simple linear relationship between l and T is postulated:

$$L(T) = \bar{l} - bT, \qquad b > 0. \tag{2}$$

The direct welfare cost of medical treatment is given by a quadratic cost:

$$\Omega(T) = \frac{a}{2}T^2, \qquad a > 0. \tag{3}$$

The diabetic's choice of diet and exercise is captured by weight (y) and consumption (c). The variation in the diabetic's weight (y) is an increasing function of consumption, and decreasing in the actual weight (captured by parameter φ). The diabetic's consumption (c) depends on the difference between his actual weight and the optimum weight ($\bar{y}$) determined by his medical adviser. In order to represent the "lulling" effect of more powerful drugs and food products, the consumption is assumed to be proportional to the medical treatment. The dynamic equations for weight and consumption are the following:

$$\dot{y} = c - \varphi y, \tag{4}$$

$$\dot{c} = T\,(\bar{y} - y)\,. \tag{5}$$

The representative diabetic takes equations (4) and (5) as constraints and maximizes the following functional:

$$\max_T \int_0^\infty U(c,T)e^{-rt}\,dt. \tag{6}$$

The first-order conditions are (after substituting equations (1)–(3) into (6))

$$H_T = 0 \Rightarrow T = \alpha\,[vb + \mu\,(\bar{y} - y)]\,, \qquad \text{where } \alpha = a^{-1}, \tag{7}$$

$$\dot{\lambda} - r\lambda = \lambda\varphi + \mu T, \tag{8}$$

$$\dot{\mu} - r\mu = -c^{-1} - \lambda. \tag{9}$$

The representative diabetic consumer takes his medication, thus $T > 0$, which implies by equation (7) that

$$T = \alpha\,[vb + \mu\,(\bar{y} - y)] > 0. \tag{10}$$

[1]One can derive equation (1) from a utility function written in terms of consumption of a composite good C, leisure X, and medical treatment T: $U(C,X,T)$. Given that the consumer allocates its time Z between work L, and leisure, and assuming that labor supply is affected by the medical treatment, we have $U(C, Z - L(T), T)$, which is equivalent to equation (1) in the text.

Equation (10) means that the diabetic balances the marginal benefits of medical treatment with its marginal costs in terms of labor supply and consumption variation.

Introducing equation (7) into (5) and (8) yields

$$\dot{c} = \alpha\left[vb + \mu\left(\bar{y} - y\right)\right]\left(\bar{y} - y\right), \tag{11}$$

$$\dot{\lambda} - r\lambda = \lambda\varphi + \mu\alpha\left[vb + \mu\left(\bar{y} - y\right)\right]. \tag{12}$$

The analysis of the complex dynamics of consumption habits is made on the system formed by equations (4), (11), (12), and (9). The steady state equilibrium $(y^*, c^*, \lambda^*, \mu^*)$ of this system is

$$\dot{c} = 0 \Rightarrow y^* = \bar{y} \Rightarrow T^* = \alpha\, vb, \qquad \text{since } T > 0, \tag{13}$$

$$\dot{y} = 0 \Rightarrow c^* = \varphi\bar{y}, \tag{14}$$

$$\dot{\lambda} = 0 \text{ and } \dot{\mu} = 0 \Rightarrow \lambda^* = -\left(\frac{r(r+\varphi)}{\alpha\, vb} + 1\right)^{-1}(c^*)^{-1} \text{ and } \mu^* = \frac{(c^*)^{-1} + \lambda^*}{r}. \tag{15}$$

This model is able to display a limit cycle between weight and consumption when three different conditions are met:

(1) when the diabetic faces large welfare costs in adapting to the medical treatment (large α, v, b);[2]
(2) the diabetic has a low rate of time preference (r);
(3) the diabetic has problem in controlling his weight (given by a low φ).

The limit cycle is stated in the proposition below.

PROPOSITION. *If $2\alpha\, vb > \varphi(r+\varphi)$, there is a limit cycle between weight and consumption.*

PROOF. According to Feichtinger *et al.* [6], in order to show the existence of a limit cycle in the optimal control model above, it is necessary to show that the signs of the determinant of the Jacobian, $|J|$, of the system (4), (11), (12), and (9), and the term K, defined below:

$$K = \begin{vmatrix} \dfrac{\partial\dot{y}}{\partial y} & \dfrac{\partial\dot{y}}{\partial\lambda} \\ \dfrac{\partial\dot{\lambda}}{\partial y} & \dfrac{\partial\dot{\lambda}}{\partial\lambda} \end{vmatrix} + \begin{vmatrix} \dfrac{\partial\dot{c}}{\partial c} & \dfrac{\partial\dot{c}}{\partial\mu} \\ \dfrac{\partial\dot{\mu}}{\partial c} & \dfrac{\partial\dot{\mu}}{\partial\mu} \end{vmatrix} + 2\begin{vmatrix} \dfrac{\partial\dot{y}}{\partial c} & \dfrac{\partial\dot{y}}{\partial\mu} \\ \dfrac{\partial\dot{\lambda}}{\partial c} & \dfrac{\partial\dot{\lambda}}{\partial\mu} \end{vmatrix} \tag{16}$$

are positive when calculated with the steady state solutions $(y^*, c^*, \lambda^*, \mu^*)$. Furthermore, the value of the bifurcation parameter given by the condition below:

$$|J| = \left(\frac{K}{2}\right)^2 + r^2\left(\frac{K}{2}\right) \tag{17}$$

must be positive as well. After simple calculation, the determinant of the Jacobian is

$$|J| = \alpha\, vb\left[r(r+\varphi) + \alpha\, vb\right] > 0, \tag{18}$$

and equation (16) yields

$$K = 2\alpha\, vb - \varphi(r+\varphi) > 0 \Leftrightarrow 2\alpha\, vb > \varphi(r+\varphi). \tag{19}$$

From equations (18) and (19), a necessary and sufficient condition for the model above to generate a limit cycle between consumption and weight is $2\alpha\, vb > \varphi(r+\varphi)$. Moreover, the bifurcation parameter α is positive and satisfies equation (17). ∎

COROLLARY. *A limit cycle between weight and consumption makes labour supply cyclical.*

PROOF. An implication of the cycle between y and c is that actual weight oscillates around $(\bar{y})$. By equation (10) this makes the medical treatment (T) fluctuate as well. However, one condition must be observed, that is, $T > 0$, which is verified only if $vb > \mu(y - \bar{y})$. Fluctuations in (T) impact on the labour supply (l) through equation (1). As a result, the diagnosed diabetic labour supply will also oscillate. ∎

[2]Note that when α rises, a and Ω fall.

3. CONCLUDING REMARKS

This paper has modelled the behaviour of a person diagnosed with diabetes. It is shown that the medical treatment of the disease creates incentives that make consumption and weight display cyclical patterns. On one hand, new powerful drugs and food products may induce the diabetic to relax his adherence to strict diet and physical exercise, thereby increasing his consumption and weight. On the other hand, if the diabetic does not follow the medical treatment, he may have health problems that have welfare costs. These health problems tend to induce a decrease in consumption and weight. The balance between these forces is at the heart of the diabetic consumption and weight cycle. One important consequence of this cycle is that labour supply will also present fluctuations.

The implications for public policy are not analysed in this paper. One can think of a specific tax (or subsidy) designed to tackle the incentives faced by diabetics in not following strict diet and physical exercise. This, however, is left for future research.

REFERENCES

1. L.P. Krall, Editor, *Joslin Diabetes Manual*, Lea & Febiger, Philadelphia, PA, (1978).
2. K. Viscusi and G. Cavallo, The effect of product safety regulation on safety precautions, *Risk Analysis* **14**, 917–929 (1994).
3. G. Feichtinger, A. Novak and F. Wirl, Limit cycles in intertemporal adjustment models, *Journal of Economic Dynamics and Control* **18**, 353–380 (1994).
4. F. Wirl and G. Feichtinger, Persistent cyclical consumption, *Rationality and Society* **7**, 156–166 (1995).
5. F. Wirl, Pathways to Hopf bifurcations in dynamic continuous-time optimization problems, *Journal of Optimization Theory and Applications* **91**, 299–320 (1996).
6. E. Dockner and G. Feichtinger, Cyclical consumption patterns and rational addiction, *American Economic Review* **83**, 256–263 (1993).